Graphing Sine and Cosine Functions

Any equation of the form $y = a\sin\left[b(x-c)\right]+d$ with $a \neq 0$ and $b \neq 0$ is a **sine function**. Its graph is called a **sine wave**, **sinusoidal wave**, or **sinusoid**. The graph of any sine function is a transformation of the graph of $y = \sin x$.

We assume x is in radians unless the problem specifically states that it is in degrees.

As the terminal side of an angle rotates around the unit circle, how does the value of the sine change?

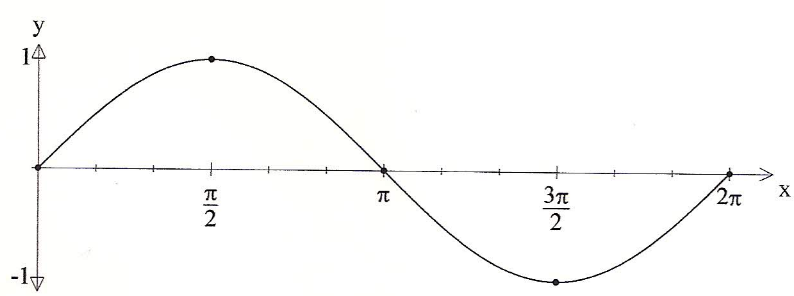
- From 0 to $\pi/2$, the sine increases from 0 to 1.
- From $\pi/2$ to π, the sine decreases from 1 to 0.
- From π to $3\pi/2$, the sine decreases from 0 to -1.
- From $3\pi/2$ to 2π, the sine increases from -1 to 0.
- The cycle repeats.

Because $\sin(x+2\pi)=\sin x$, the shape we see in the interval $[0,2\pi]$ repeats on the intervals $[2\pi,4\pi],[4\pi,6\pi],[-2\pi,0],[-4\pi,-2\pi]$, etc.

A repeating function like $y=\sin x$ is called a **periodic function.** The length of the smallest non-repeating unit is the **period** of the function. The period of $y=\sin x$ is 2π. The graph of $y=\sin x$ over any interval of length 2π is called a **cycle**. The graph of $y=\sin x$ over $[0,2\pi]$ is the **fundamental cycle**.

Key points on the graph of $y=\sin x$:

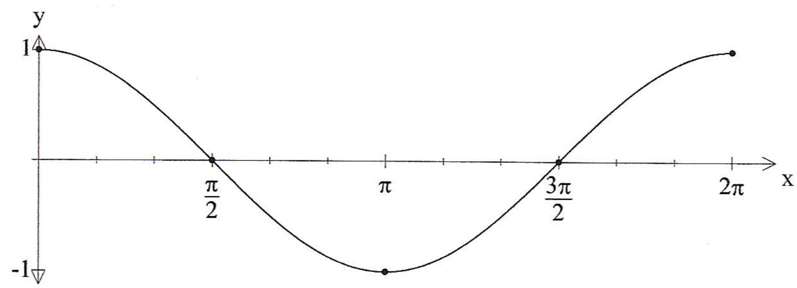
x	0	$\pi/2$	π	$3\pi/2$	2π
$y=\sin x$	0	1	0	-1	0



The graph of $y=\cos x$ has the same shape as the graph of $y=\sin x$, but it is shifted to the left by a distance of $\pi/2$. For this reason, the graph of $y=\cos x$ is also called a sine wave. The graph of $y=\cos x$ over $[0,2\pi]$ is called the **fundamental cycle** of $y=\cos x$.

Key points on the graph of $y = \cos x$:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \cos x$	1	0	-1	0	1

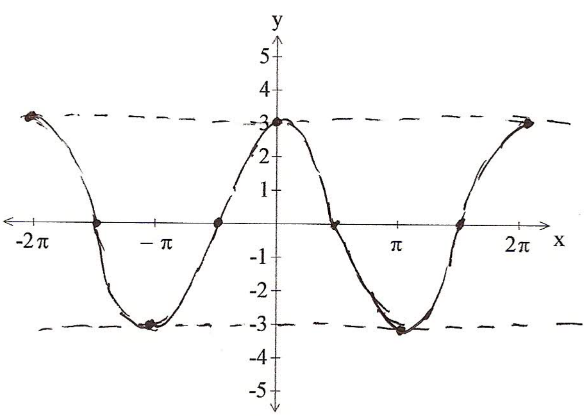


The effect of changing the value of *a*:

The **amplitude** of $y = a\sin x$ or $y = a\cos x$ is $|a|$. The amplitude is the "height" of the sine wave. It is half the difference between the maximum and minimum points on the graph. If a is negative, the graph is reflected over the x-axis.

Examples: Sketch the graphs of the following and determine the amplitude and range of each.

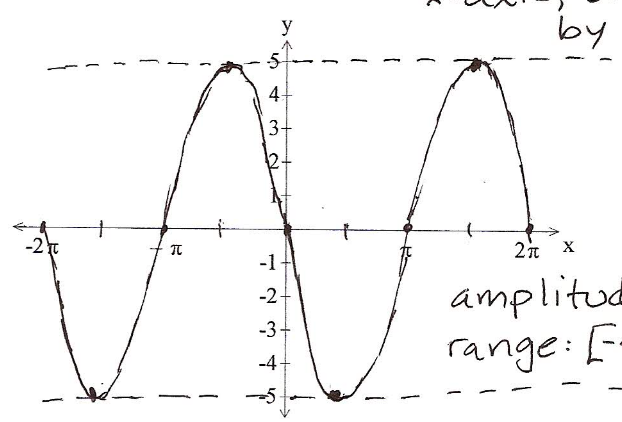
$y = 3\cos x$



amplitude: 3

range: [-3, 3]

$y = -5\sin x$ ← reflected over x-axis, stretched by 5



amplitude: 5

range: [-5, 5]

x	0	π/2	π	3π/2	2π
y = sin x	0	1	0	-1	0
y = -5 sin x	0	-5	0	5	0

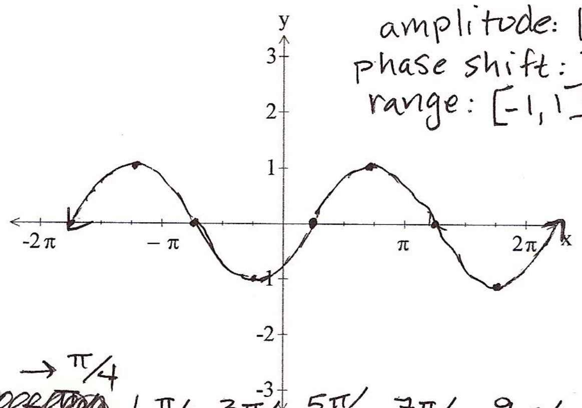
The effect of changing the value of *c*:

The **phase shift** of the graph of $y = \sin(x - c)$ or $y = \cos(x - c)$ is *c*. This means that the graph is shifted *c* units to the right if *c* is positive, or *c* units to the left if *c* is negative.

Examples: Sketch the graphs of the following, and find the amplitude, phase shift, and range of each.

$y = \sin(x - \pi/4)$ shift → $\pi/4$

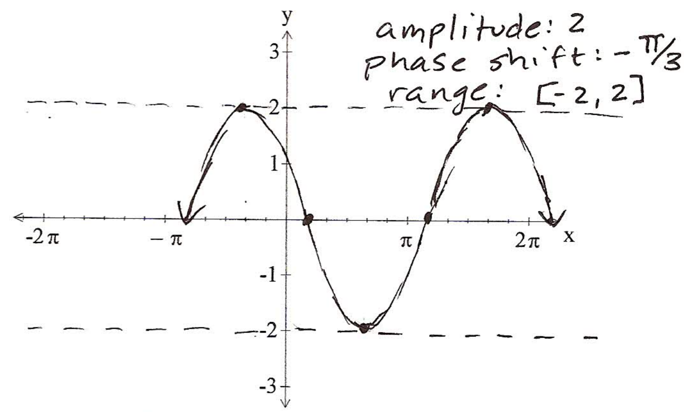
amplitude: 1
phase shift: $\pi/4$
range: $[-1, 1]$



→ $\pi/4$	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$	$9\pi/4$
x	0	$\pi/2$	π	$3\pi/2$	2π
y = sin x	0	1	0	-1	0

$y = 2\cos(x + \pi/3)$ stretch by 2, shift ← $\pi/3$

amplitude: 2
phase shift: $-\pi/3$
range: $[-2, 2]$



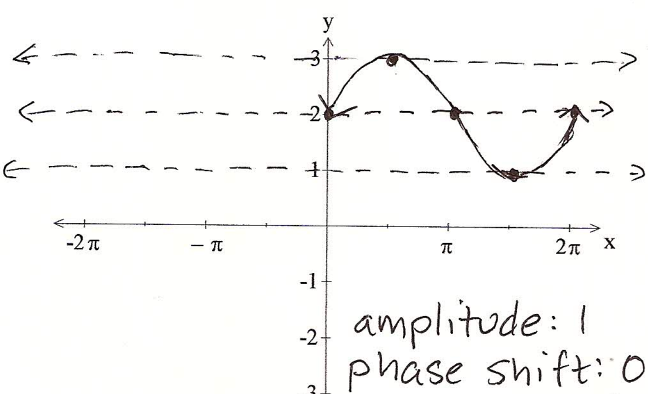
← $\pi/3$	$-\pi/3$	$\pi/6$	$2\pi/3$	$7\pi/6$	$5\pi/3$
x	0	$\pi/2$	π	$3\pi/2$	2π
y = cos x	1	0	-1	0	1
stretch by 2	2	0	-2	0	2

The effect of changing the value of *d*:

The **vertical translation** of the graph of $y = \sin x + d$ or $y = \cos x + d$ is *d*. This means that the graph is shifted *d* units up if *d* is positive, or *d* units down if *d* is negative.

Examples: Sketch the graphs of the following, and find the amplitude, phase shift, vertical shift, and range of each.

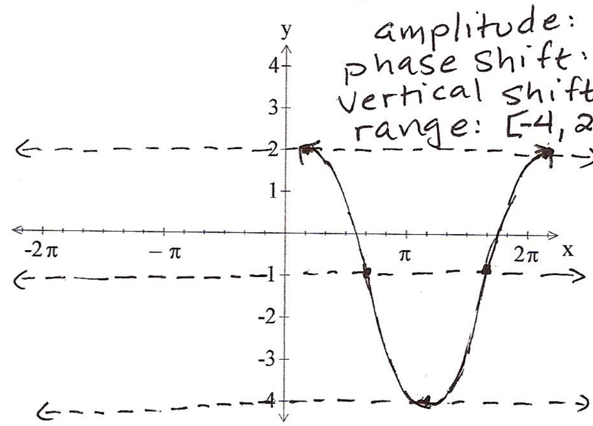
$y = \sin x + 2$ ↑ 2



amplitude: 1
phase shift: 0
vertical shift: 2
range: $[1, 3]$

$y = 3\cos(x - \pi/6) - 1$ stretch by 3, → $\pi/6$, ↓ 1

amplitude: 3
phase shift: $\pi/6$
vertical shift: -1
range: $[-4, 2]$



→ $\pi/6$	$\pi/6$	$2\pi/3$	$7\pi/6$	$5\pi/3$	$13\pi/6$
x	0	$\pi/2$	π	$3\pi/2$	2π
y = cos x	1	0	-1	0	1
stretch by 3	3	0	-3	0	3
↓ 1	2	-1	-4	-1	2

Examples: Find the equation of each sine wave in its final position.

1. The graph of $y=\sin x$ is stretched by a factor of 2, reflected in the x-axis, shifted $\pi/5$ units to the right, then translated 4 units downward.

$y=\sin x$

1. stretch by 2: $y=2\sin x$
2. reflect in x-axis: $y=-2\sin x$
3. shift $\pi/5$ to right: $y=-2\sin(x-\pi/5)$
4. shift down 4: $\boxed{y=-2\sin(x-\pi/5)-4}$

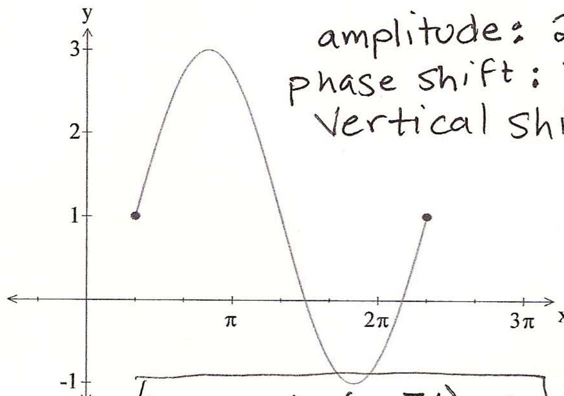
2. The graph of $y=\cos x$ is shifted $\pi/3$ units to the left, translated upward 2 units, then stretched by a factor of 2.

1. shift $\pi/3$ left: $y=\cos(x+\pi/3)$
2. translate up 2: $y=\cos(x+\pi/3)+2$
3. stretch by 2: $y=2[\cos(x+\pi/3)+2]$

$\boxed{y=2\cos(x+\pi/3)+4}$

Examples: Find an equation of the requested form whose graph is the given sine wave.

$y=a\sin(x-c)+d$

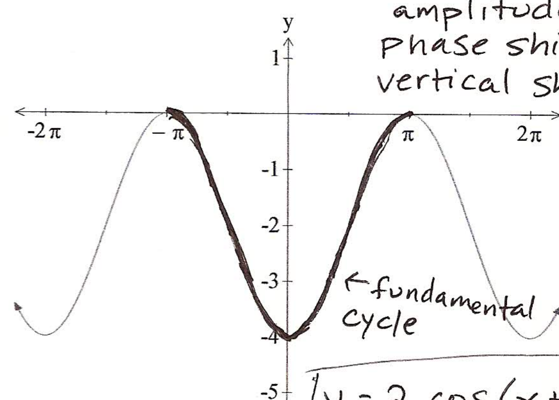


amplitude: 2
phase shift: $\pi/3$
vertical shift: 1

$\boxed{y=2\sin(x-\pi/3)+1}$

$y=a\cos(x-c)+d$

amplitude: 2
phase shift: $-\pi$
vertical shift: -2



$\boxed{y=2\cos(x+\pi)-2}$

or

$\boxed{y=-2\cos x-2}$

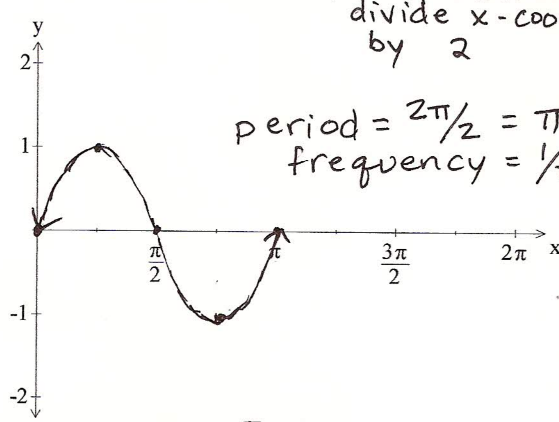
The effect of changing the value of *b*:

The **period** of the graph of $y=\sin(bx)$ or $y=\cos(bx)$ for $b>0$ is given by $P=2\pi/b$. This means that there are b cycles every 2π units. The **frequency**, F, of a sine wave with period P is defined by $F=1/P$.

Examples: Sketch the graphs of the following and determine the period and frequency of each.

$y=\sin(2x)$ horizontal compression: divide x-coords. by 2

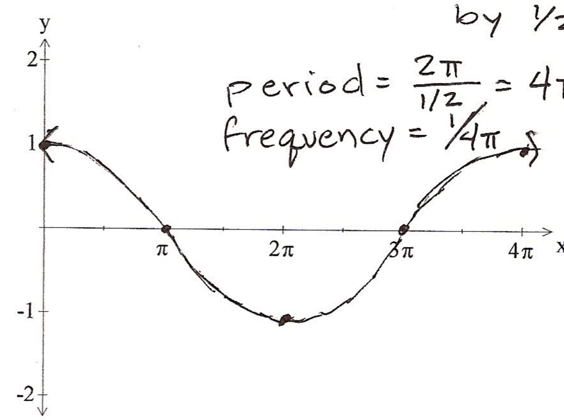
period $=2\pi/2=\pi$
frequency $=1/\pi$



horiz. comp.	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	0	$\pi/2$	π	$3\pi/2$	2π
y = sin x	0	1	0	−1	0

$y=\cos(x/2)$ horizontal stretch: divide x-coords. by 1/2

period $=\frac{2\pi}{1/2}=4\pi$
frequency $=1/4\pi$



horiz. stretch	0	π	2π	3π	4π
x	0	$\pi/2$	π	$3\pi/2$	2π
y = cos x	1	0	−1	0	1

The general sine wave:

Characteristics of the graph of $y = a\sin\left[b(x-c)\right]+d$ or $y = a\cos\left[b(x-c)\right]+d$:

- Amplitude: $|a|$
- Period: $P = 2\pi/b$
- Frequency: $F = 1/P = b/2\pi$
- Phase shift: c
    - Shift right for $c > 0$ – i.e., the equation has the term $(x-c)$.
    - Shift left for $c < 0$ – i.e., the equation has the term $(x+c)$.
- Vertical translation: d
    - Shift up for $d > 0$.
    - Shift down for $d < 0$.

Steps to graph $y = a\sin\left[b(x-c)\right]+d$ or $y = a\cos\left[b(x-c)\right]+d$:

1. Start with the five key points on the graph of $y = \sin x$ or $y = \cos x$.
2. Find five key points for $y = a\sin\left[b(x-c)\right]+d$ or $y = a\cos\left[b(x-c)\right]+d$ by
    a. dividing each x-coordinate by b and adding c.
    b. multiplying each y-coordinate by a and adding d.
    c. sketch one cycle of your graph through the five new points.

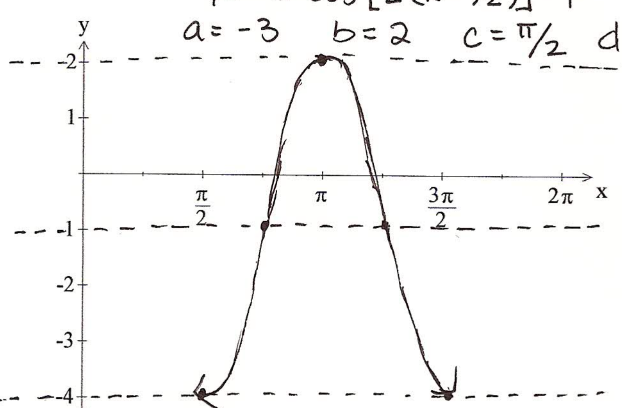
***Note:** Order is important. Multiply or divide first, then add.

Examples: Determine the amplitude, period, frequency, phase shift, and vertical shift of the following. Then sketch one cycle of each graph.

$y = -3\cos(2x-\pi)-1$

$y = -3\cos[2(x-\pi/2)]-1$

$a = -3 \quad b = 2 \quad c = \pi/2 \quad d = -1$



amplitude: 3 period: $2\pi/2 = \pi$

frequency: $1/\pi$ phase shift: $\pi/2$

vertical shift: -1

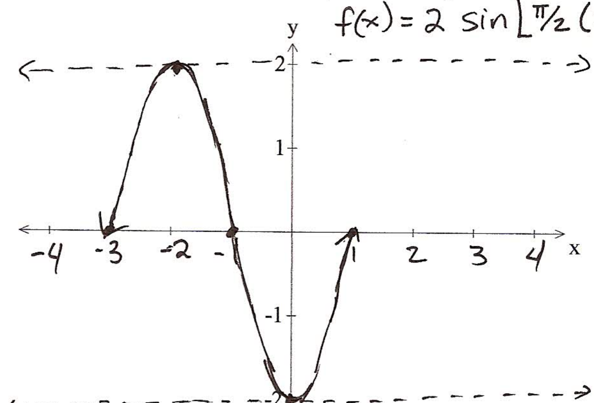
$\to \pi/2$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$
horiz. comp	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \cos x$	1	0	-1	0	1
stretch & reflect	-3	0	3	0	-3
$\downarrow 1$	-4	-1	2	-1	-4

$f(x) = 2\sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}\right)$

$a = 2 \quad b = \pi/2$

$c = -3 \quad d = 0$

$f(x) = 2\sin[\pi/2(x+3)]$



amplitude: 2 period: $\frac{2\pi}{\pi/2} = 4$

frequency: $1/4$ phase shift: -3

$\leftarrow 3$	-3	-2	-1	0	1
horiz. comp.	0	1	2	3	4
x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \sin x$	0	1	0	-1	0
vert. stretch	0	2	0	-2	0

Graphing Secant and Cosecant Functions

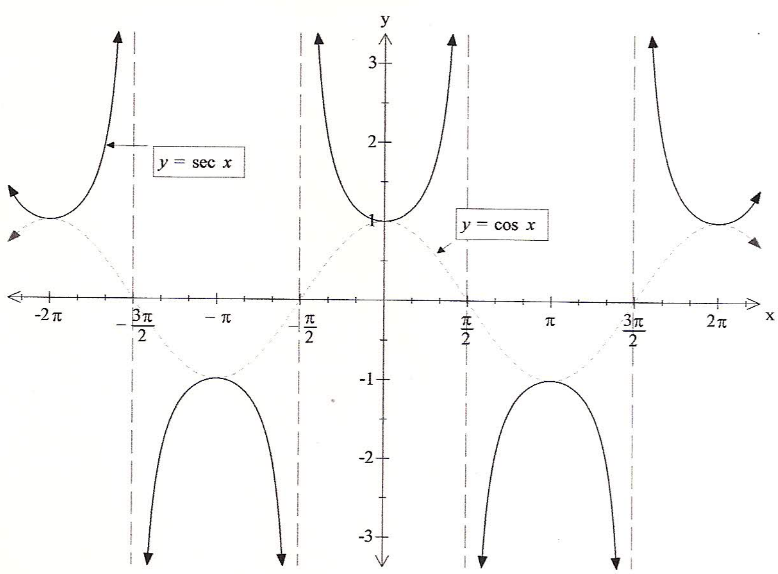
- Remember, $\sec x = 1/\cos(x)$ and $\csc x = 1/\sin(x)$.
  - We aren't allowed to divide by 0. This means:
    - Whenever $\cos x = 0$, $\sec x$ is undefined and whenever $\sin x = 0$, $\csc x$ is undefined.
      - Places where $\cos x = 0$ and $\sec x$ is undefined:
        $\pi/2$, $3\pi/2$, $5\pi/2$, $7\pi/2$, etc. $x = \pi/2 + k\pi$, k is an integer
      - Places where $\sin x = 0$ and $\csc x$ is undefined:
        0, π, 2π, 3π, etc. $x = k\pi$, k is an integer
    - The graphs of $y = \sec x$ and $y = \csc x$ have vertical asymptotes at these locations.
    - **To Find the Equations of the Asymptotes:**
      - Find the smallest non-negative x where the function is undefined.
      - Add this value to k times the distance between the asymptotes.

Graphing Secant Functions

- A secant function has the form $y = a\sec\left[b(x-c)\right]+d$.

- The domain of $y = \sec x$ is the set of all real numbers except those of the form $\pi/2 + k\pi$, where k is an integer. The equations of the vertical asymptotes are $x = \pi/2 + k\pi$ for any integer k. The period is 2π. The range is $(-\infty,-1]\cup[1,\infty)$.

- To graph $y = a\sec\left[b(x-c)\right]+d$:
  - Sketch the graph of $y = a\cos\left[b(x-c)\right]+d$.
  - Wherever the graph of the cosine function crosses the x-axis, draw a vertical asymptote.
  - The local maxima of the graph of the cosine function become local minima on the graph of the secant function with $y \to \infty$ as x approaches the asymptotes on either side. The local minima of the graph of the cosine function become local maxima on the graph of the secant function with $y \to -\infty$ as x approaches the asymptotes on either side.



Key Points:

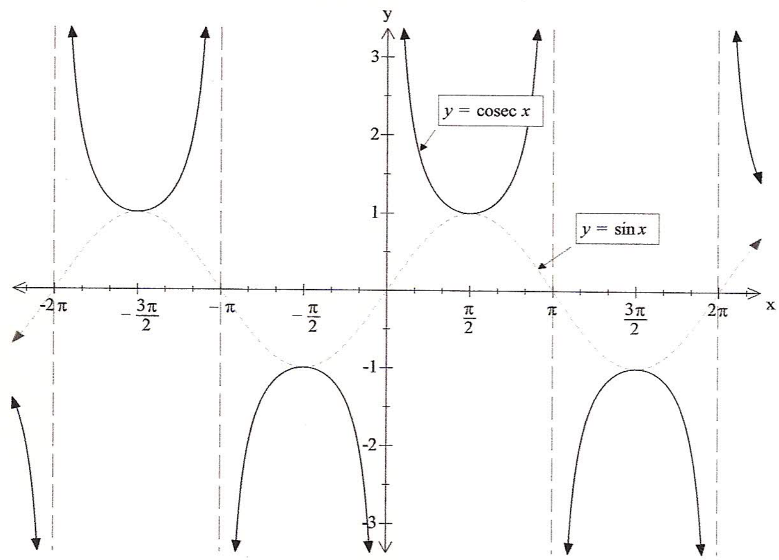
x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \sec x$	1	undef	-1	undef	1

Graphing Cosecant Functions

- A cosecant function has the form $y = a\csc\left[b\left(x-c\right)\right]+d$.

- The domain of $y=\csc x$ is the set of all real numbers except those of the form $k\pi$, where k is an integer. The equations of the vertical asymptotes are $x = k\pi$ for any integer k. The period is 2π. The range is $(-\infty,-1]\cup[1,\infty)$.

- To graph $y = a\csc\left[b\left(x-c\right)\right]+d$:
  - Sketch the graph of $y = a\sin\left[b\left(x-c\right)\right]+d$.
  - Wherever the graph of the sine function crosses the x-axis, draw a vertical asymptote.
  - The local maxima of the graph of the sine function become local minima on the graph of the cosecant function with $y\to\infty$ as x approaches the asymptotes on either side. The local minima of the graph of the sine function become local maxima on the graph of the cosecant function with $y\to-\infty$ as x approaches the asymptotes on either side.



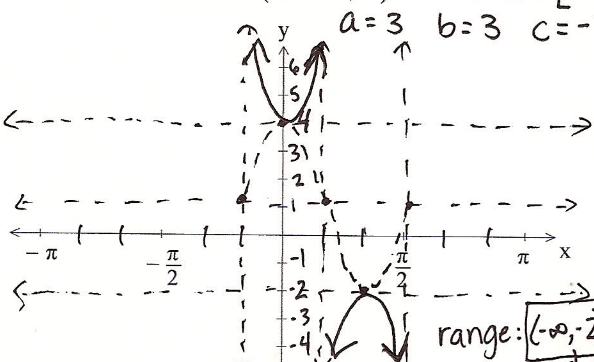
Key Points:

x	0	$\pi/2$	π	$3\pi/2$	2π
y	undef	1	undef	-1	undef

Examples: Graph the following functions. Find the period, asymptotes, and range of each.

$y=3\csc(3x+\pi/2)+1 = 3\csc\left[3(x+\pi/6)\right]+1$

$a=3 \quad b=3 \quad c=-\pi/6 \quad d=1$



range: $\boxed{(-\infty,-2]\cup[4,\infty)}$

period $= \frac{2\pi}{b}$

period $= \boxed{\frac{2\pi}{3}}$

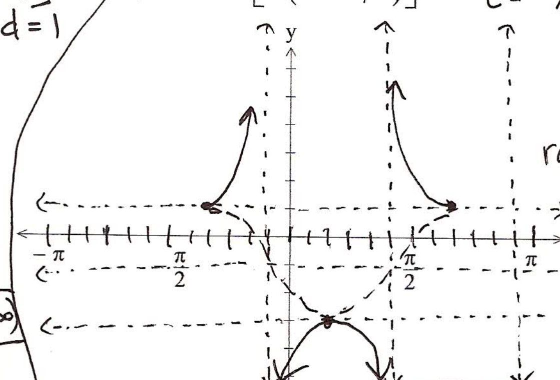
asymptotes are $\pi/3$ units apart

$\boxed{x = \pi/6 + k\pi/3}$

← $\pi/6$	$-\pi/6$	0	$\pi/6$	$\pi/3$	$\pi/2$
horiz. comp.	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$
x	0	$\pi/2$	π	$3\pi/2$	2π
y	undef	1	undef	-1	undef
v. stretch	undef	3	undef	-3	undef
↑ 1	undef	4	undef	-2	undef

$y=2\sec\left[2\left(x+\pi/3\right)\right]-1$

$a=2 \quad b=2$
$c=-\pi/3 \quad d=-1$



range: $\boxed{(-\infty,-3]\cup[1,\infty)}$

period $= \frac{2\pi}{b}$

period $= \frac{2\pi}{2}$

$= \boxed{\pi}$

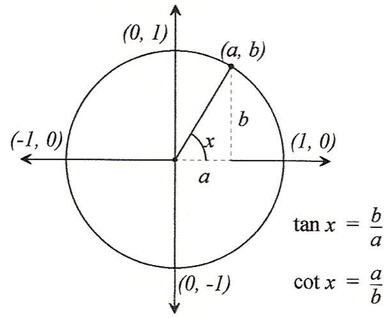
asymptotes are $\pi/2$ units apart

$\boxed{x = 5\pi/12 + k\pi/2}$

← $\pi/3$	$-\pi/3$	$-\pi/12$	$\pi/6$	$5\pi/12$	$2\pi/3$
horiz. comp.	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	0	$\pi/2$	π	$3\pi/2$	2π
y	1	undef	-1	undef	1
v. stretch	2	undef	-2	undef	2
↓ 1	1	undef	-3	undef	1

Graphing Tangent and Cotangent Functions

Let (a,b) be coordinates of points on the unit circle. For any given angle x, $\tan x = b/a$. This means that $y = \tan x$ is undefined whenever $a = 0$. For any given angle x, $\cot x = a/b$. This means that $y = \cot x$ is undefined whenever $b = 0$.

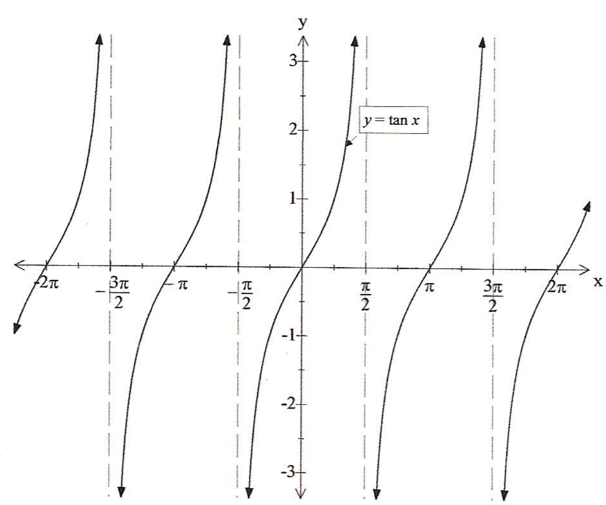


Graphing Tangent Functions:

The domain of $y = \tan x$ is the set of all real numbers except numbers of the form $\pi/2 + k\pi$, where k is an integer. The equations of the vertical asymptotes are $x = \pi/2 + k\pi$, where k is an integer.

Five key points on the graph of $y = \tan x$ define the fundamental cycle:

x	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
$y = \tan x$	undefined	-1	0	1	undefined



To graph $y = a\tan\left[b(x-c)\right] + d$:

1. Start with the three key points on the graph of $y = \tan x$ and the equations of the asymptotes.
2. Find three key points and the asymptotes for $y = a\tan\left[b(x-c)\right] + d$ by:
   a. dividing each x-coordinate by b and adding c. (Treat the equations of the asymptotes like x-coordinates.)
   b. multiplying each y-coordinate by a and adding d.
3. Sketch one cycle of $y = a\tan\left[b(x-c)\right] + d$ through the three new points and approaching the new asymptotes.

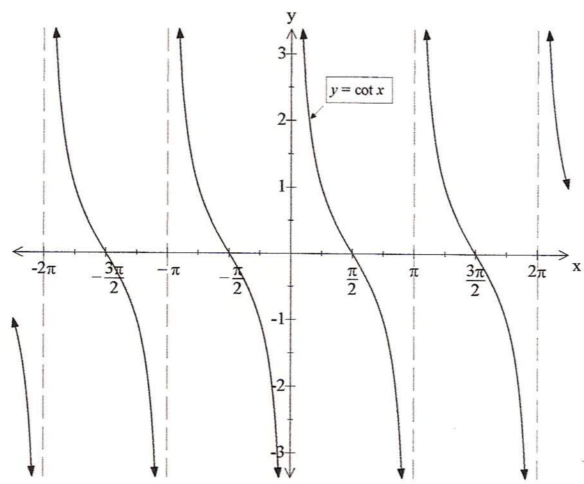
Note: The period of $y = a\tan\left[b(x-c)\right] + d$ and $y = a\cot\left[b(x-c)\right] + d$ is π/b rather than $2\pi/b$.

Graphing Cotangent Functions:

The domain of $y = \cot x$ is the set of all real numbers except numbers of the form $k\pi$, where k is an integer. The equations of the vertical asymptotes are $x = k\pi$, where k is an integer.

Five key points on the graph of $y = \cot x$ define the fundamental cycle:

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$y = \cot x$	undefined	1	0	-1	undefined



To graph $y = a\cot\left[b(x-c)\right]+d$:

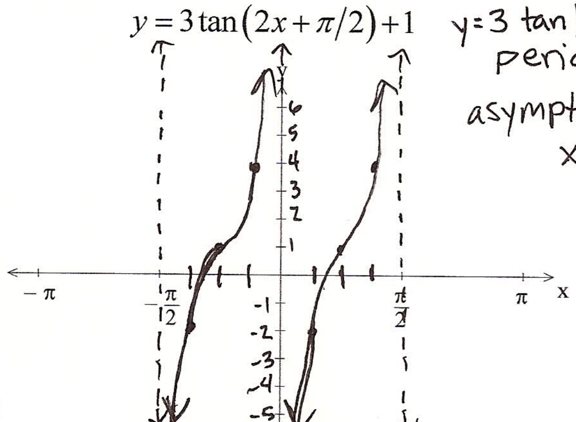
4. Start with the three key points on the graph of $y = \cot x$ and the equations of the asymptotes.
5. Find three key points and the asymptotes for $y = a\cot\left[b(x-c)\right]+d$ by:
    a. dividing each x-coordinate by b and adding c. (Treat the equations of the asymptotes like x-coordinates.)
    b. multiplying each y-coordinate by a and adding d.
6. Sketch one cycle of $y = a\cot\left[b(x-c)\right]+d$ through the three new points and approaching the new asymptotes.

Examples: Graph the following functions. Find the period and the equations of the asymptotes of each.

$y = 3\tan(2x + \pi/2) + 1$ $\quad y = 3\tan\left[2(x + \pi/4)\right] + 1$

period $= \pi/2$

asymptotes: $x = k\pi/2$

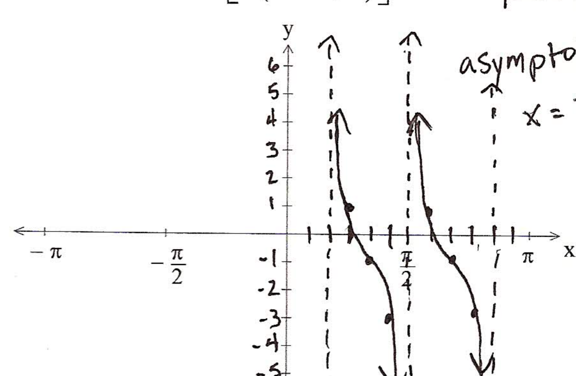


$\leftarrow \pi/4$	$-\pi/2$	$-3\pi/8$	$-\pi/4$	$-\pi/8$	0
h. compression by 2	$-\pi/4$	$-\pi/8$	0	$\pi/8$	$\pi/4$
x	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
y	und	-1	0	1	und
v. stretch	und	-3	0	3	und
$\uparrow 1$	und	-2	1	4	und

$y = 2\cot\left[3(x - \pi/6)\right] - 1$

period $= \pi/3$

asymptotes: $x = \pi/6 + k\pi/3$



$\rightarrow \pi/6$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
h. compression	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$
x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
y	und	1	0	-1	und
v. stretch	und	2	0	-2	und
$\downarrow 1$	und	1	-1	-3	und